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ON THE STATUS OF ST.-VENANT'S SOLUTIONS AS MINIMIZERS OF ENERGY

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Abstract—Solutions of the type introduced by St.-Venant for linearly elastic prisms have some limited status as solutions of minimum energy although, commonly, there are other solutions with lower energy. Our purpose is to clarify this.

1. INTRODUCTION

In 1966, Sternberg and Knowles[1] explored the possibility that the St.-Venant solutions for torsion, bending, etc. minimize the strain energy, among solutions for prisms with free sides, yielding the same resultant forces and moments. They established some theorems of this kind, for homogeneous, isotropic materials. However, this minimization occurs only if the solutions involved in the comparison are subject to additional, quite restrictive conditions. We refer the reader to their paper for details. Later, Maisonneuve[2] characterized the solutions which do minimize the energy. They obtain from solving a boundary value problem wherein the displacement is prescribed to be zero on one end, to have the form of an infinitesimal rigid displacement at the other. Of course, the translation and rotation involved must be adjusted to give desired resultants, which is feasible. Also, to Maisonneuve's solutions, we can add the usual trivial rigid displacement; his boundary conditions normalize the choice in a particular way. The St.-Venant solution for torsion of a circular prism made of homogeneous isotropic material satisfies such boundary conditions, but it is easy to see that most of the St.-Venant solutions do not.

We don't find fault with these analyses. However, rather commonly, there is good intuitive reason to exclude from consideration many solutions. If such considerations exclude those characterized by Maisonneuve, some other solution can become the minimizer, in the set allowed. One such restriction, which is plausible in many particular situations, leaves some solutions of St.-Venant type as minimizers. Our purpose is to elaborate this.

Consider common tensile tests, for example. Usually, the specimen is not really a prism, but a central part is, ends being shaped to accomodate grips. With St.-Venant, we must agree that the details of loading on the nominal ends of the prismatic part are not known. However, very often, the symmetry of the specimen and loading device are such that, if we were given the load distribution on one end, or a suitable part of it, we could reasonably infer the distribution on the opposite end. Similar remarks apply to other physical situations, such as cases of combined tension and torsion, or bending. In such cases, the St.-Venant solutions often meet the symmetry requirements, and it makes sense to compare them only with other solutions exhibiting this symmetry. A theorem given below covers such a type of symmetry.

2. SOLUTIONS OF ST.-VENANT TYPE

Consider a prism. Referred to rectangular Cartesian coordinates, with the x_3 axis parallel to its generators, it will occupy a region of the form

$$D\mathrm{x}[0, L], \tag{1}$$

where D is some region in the x_1–x_2 plane, and L is the length of the prism. We presume constitutive equations of the form used in linear elasticity theory, viz

$$2W = A_{ijkl}\epsilon_{ij}\epsilon_{kl}, \tag{2}$$

$$t_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} = \frac{\partial W}{\partial u_{i,j}}, \tag{3}$$

where W is the strain energy per unit volume, t_{ij} and ϵ_{ij} are the stress and strain components, related to the displacements u_i by

$$2\epsilon_{ij} = u_{i,j} + u_{j,i}. \tag{4}$$

As usual, we assume that

$$W > 0 \text{ if } \epsilon_{ij} \neq 0. \tag{5}$$

All solutions considered must satisfy the equilibrium equations

$$t_{ij,j} = 0 \text{ in } Dx[0, L], \tag{6}$$

and free boundary conditions on the lateral surface

$$t_{ij}\nu_j = 0 \text{ on } \partial Dx[0, L], \tag{7}$$

where ν is the unit normal. Here, Latin indices take on values 1–3. Below, Greek indices will take on values 1 and 2. With customary assumptions of smoothness of D and the solutions, the traction vectors $t_{i3}(x_\alpha, L)$ and $-t_{i3}(x_\alpha, 0)$ on the ends will give balancing forces and moments, as indicated by

$$\int_D t_{i3}\Big|_0^L = 0, \tag{8}$$

$$\int_D (x_1 t_{23} - x_2 t_{13})\Big|_0^L = 0, \tag{9}$$

$$\int_D x_\alpha t_{33}\Big|_0^L = L\int_D t_{\alpha 3}(x_\beta, L). \tag{10}$$

By a *solution of St.-Venant type*, we shall mean one which results from the semi-inverse assumption that the strain is independent of x_3,

$$\epsilon_{ij} = \epsilon_{ij}(x_\alpha). \tag{11}$$

St.-Venant's solutions for torsion, tension and bending are of this type, but his flexure solutions are not. Similar kinds of solutions can be expected to occur for materials which are anisotropic and inhomogeneous, as long as the A's occurring in (2) are independent of x_3, and are reasonably smooth functions of x_1 and x_2. For homogeneous, anisotropic materials, pertinent analyses are given by Lekhnitskii[3]. It is a relatively simple problem in kinematics to characterize the displacement fields which are compatible with (11). He ([3], eqn 17.12) gives the answer. Like other writers, he takes as primitive the assumption that the stresses are independent of x_3. For materials of the type indicated above, this is equivalent, but one must check to see which analyses are purely kinematical, and which make some use of equilibrium equations, etc. His equation ([3], 17.12) seems to involve elastic moduli, but they can be eliminated by redefining arbitrary constants. In any even., such displacements have the form

$$u_i(x_\alpha, x_3) = u_i(x_\alpha, 0) + x_3(a_i + b_i x_3 + \epsilon_{ijk} c_j x_k), \tag{12}$$

where the functions $u_i(x_\alpha, 0)$ are arbitrary. The constant vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ must satisfy the following conditions:

$$2b_1 + c_2 = 2b_2 - c_1 = b_3 = 0. \tag{13}$$

An analogous result for finite deformations is given by Ericksen ([4], Section 3). Some of the mechanics there treated is not well-known in linear theory, although obvious adaptations do apply.

Let

$$\left.\begin{aligned} \alpha_i &= (a_i + b_i L)L, \\ \beta_i &= c_i L. \end{aligned}\right\} \tag{14}$$

Then

$$v_i = \alpha_i + \epsilon_{ijk}\beta_j x_k \ . \tag{15}$$

clearly represents a rigid displacement, and we have

$$u_i(x_\alpha, L) - u_i(x_\alpha, 0) = v_i(x_\alpha, L), \tag{16}$$

a fact which plays an important role in later analysis. When (12) applies, Maisonneuve's solutions will satisfy (16) with $u_i(x_\alpha, 0) = 0$, and some choice of constants α_i and β_i, but, in general, will not be of the form (12). Of course, solutions of the St.-Venant type will involve a subset of (12), those which also satisfy (6) and (7), for some material of the type indicated above. For these, the stresses will be independent of x_3, so (10) requires that the resultant force be normal to the end,

$$\int_D t_{\alpha 3} = 0. \tag{17}$$

Of course, it is for this reason that one needs some different type of solution for the problems of flexure, even if we accept St.-Venant's Principle. On the other hand, we do expect to have solutions which are useful for describing tension, bending, or torsion.

3. MINIMIZATION

For the kinds of problems for which we are likely to consider using the solutions of St.-Venant type described above, it seems to us probable that, rather commonly, symmetry considerations will suggest that the actual load distribution will conform to the equation

$$t_{i3}(x_\alpha, L) = t_{i3}(x_\alpha, 0). \tag{18}$$

Again, this assumption implies (17), so it can hardly be appropriate for problems of flexure. Put in physical terms, we must decide whether tractions applied at opposite ends of a generator are equal in magnitude and opposite in sign. If so, (18) applies. Let bars denote any solution compatible with (18), giving rise to the same resultant force and moment as a solution of the St.-Venant type, with displacement u_i.

Then

$$\bar{\bar{u}} = \bar{u}_i - u_i \tag{19}$$

will give a solution with zero resultants. Calculating the total energy E, the integral of W over the body, we will have, in obvious notation,

$$\bar{E} = E + \bar{\bar{E}} + \int_{Dx[0,L]} \bar{\bar{t}}_{ij} u_{i,j}, \tag{20}$$

$$Dx[0, L)$$

by using elementary properties of W which follow from (2) and (3).

We now employ (6), (7), (15) and (16) as well as the fact that $\bar{\bar{t}}_{i3}$ confirms to (18), to calculate that

$$\int_{Dx[0,L]} \bar{\bar{t}}_{ij} u_{i,j} = \int_D \bar{\bar{t}}_{i3} u_i \Big|_0^L = \alpha_i \int_D \bar{\bar{t}}_{i3}(x_\alpha, L) + \beta_i \int_D \epsilon_{ijh} x_j \bar{\bar{t}}_{h3}(x_\alpha \Big|_{x_3 = L} . \tag{21}$$

However, since $\bar{u}_i$ is a solution with zero resultant force and moment, these integrals vanish, so

$$\bar{E} - E = \hat{E} \geqq 0. \tag{22}$$

Furthermore, from the time of Kirchhoff, it has been familiar that the energy of a linearly elastic solution can vanish only if it is a trivial solution, representing a rigid displacement, a consequence of (5). Thus the inequality is strict, unless $\bar{u}_i$ and u_i are so related. In this sense, then, the St.-Venant solutions do minimize the energy, among the solutions which conform to (18) and have matching resultants.

Incidentally, our analysis also gives a quick proof that two solutions of the St.-Venant type, with matching resultants, are trivially related. Of course, this can be established by other methods, and the observation is hardly new. The analysis leading to Maisonneuve's results, is much the same. In (21), the integral involved also vanishes if $u_i = 0$ at one end and, at the other, u_i has the form of a rigid displacement, whether or not (18) applies. Neither is it important that the elastic moduli be independent of x_3. From the two results, such solutions satisfy (18) if and only if they are of St.-Venant type, when the moduli are independent of x_3.

Of course, one might consider different restrictions, as alternatives to (18), which are suggested by other notions of symmetry. Sometimes, for example, we have reason to think that loads on one end will display some symmetry with respect to reflections. If such restrictions imply (18), and do not exclude solutions of the St.-Venant type, they will, of course, continue to be minimizers. With respect to some of the more likely and obvious symmetries of this kind, solutions of the St.-Venant type seem to easily accept the limitations. Some types of flexure problems do involve rather obvious symmetries, as can be seen by considering a beam with weights hung from each end, suitably supported near its center. We have not given serious thought to such problems, but the need to account for central supporting clearly complicates analysis.

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